Physics 53

Rotational Motion 1

We're going to turn this team around 360 degrees.
— Jason Kidd

Rigid bodies

To a good approximation, a solid object behaves like a perfectly **rigid body**, in which each particle maintains a fixed spatial relationship to the other particles. This is an approximation, because in a real object the atoms actually oscillate about their average "equilibrium" positions in thermal motions. Here we will ignore these oscillations.

The motion of a rigid body, like that of any system of particles, consists of two parts: motion of the CM and motion relative to the CM. For a rigid body the latter motion consists entirely of **rotation**, with each particle moving in a circle about some point on an axis of rotation passing through the CM. The circles described by the particles of a rigid body do not all lie in the same plane (although they are in parallel planes). We must therefore generalize our description of circular motion to three dimensions.

Vectors in circular motion

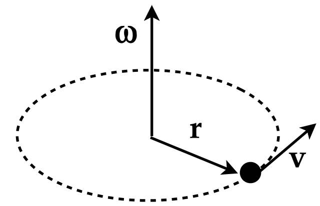
One new feature in the description is that the angular velocity becomes a three-dimensional *vector*, denoted by $\boldsymbol{\omega}$. It is oriented perpendicular to the plane of the circle, in a direction given by a **right-hand rule**:

Direction of angular velocity	The vector $\boldsymbol{\omega}$ is perpendicular to the plane of the circle followed by the particle. Curl the fingers of the right hand the way the particle moves around the circle. The thumb points in the direction of $\boldsymbol{\omega}$.

This "right hand rule" is the first of many in physics; most of them arise from the properties of the vector product, to be discussed in the next section.

In our earlier discussion of circular motion we had a the relation $v = r\omega$ between the linear and angular speeds. Now we will show the correct relation among the vectors.

The position vector **r** and the velocity vector **v** lie in the plane of the circle. The situation is as shown. Note that the three vectors are mutually perpendicular.



The vector product

The *vector* relationship among **r**, **v** and **ω** involves a new quantity, the product of two vectors which yields another vector.

The **vector product** of vectors **A** and **B** is written $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. (Because of the notation it is often called the "cross" product.) Its definition in terms of components is as follows:

$$\begin{aligned} C_x &= A_y B_z - A_z B_y \\ C_y &= A_z B_x - A_x B_z \\ C_z &= A_x B_y - A_y B_x \end{aligned}.$$

It is usually more practical to give rules, in arrow representation, for the magnitude and direction of the result. Let the arrows representing **A** and **B** be placed tail to tail, so that the angle between them is θ. Then we have:

Rules for vector product: $\mathbf{C} = \mathbf{A} \times \mathbf{B}$	Magnitude: $C = AB\sin\theta$. Direction: C perpendicular to A and B. Curl fingers of right hand from A to B. Thumb points in direction of C.

The direction rule is the basis of most of the right-hand rules in physics.

Some important properties of the vector product:

- If $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then **C** is perpendicular to the plane containing **A** and **B**, and therefore perpendicular to both **A** and **B**.
- The order of the vectors is important, since $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$.
- If the two vectors are along the same line their vector product is zero ($\sin\theta = 0$).
- If the two vectors are perpendicular, the magnitude of the vector product is the product of their magnitudes ($\sin\theta = 1$). This case has the largest possible magnitude.

Since $A\sin\theta$ is the component of **A** perpendicular to **B**, and also $B\sin\theta$ is the component of **B** perpendicular to **A**, a useful way to think of the magnitude is this:

The magnitude of the vector product is equal to the magnitude of either vector multiplied by the component of the other vector perpendicular to the first.

Circular motion vectors

We can use the vector product to rewrite our equations for circular motion. For the velocity of the particle we have

Velocity in circular motion	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

The time derivative of this product gives rise to the radial and tangential accelerations we discussed earlier in Kinematics 3. In vector form we have:

Radial acceleration	$\mathbf{a}_r = -\omega^2 \mathbf{r}$

Tangential acceleration	$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$

Here $\boldsymbol{\alpha} = d\boldsymbol{\omega} / dt$ is the angular acceleration vector. It is parallel to $\boldsymbol{\omega}$ if the speed of rotation is increasing, antiparallel to $\boldsymbol{\omega}$ if the speed is decreasing.

The magnitudes of these vectors obey same formulas we had in our earlier discussion.

Kinetic energy of a particle in circular motion

For a single particle executing circular motion, the kinetic energy is, using $v = \omega r$:

$$K = \tfrac{1}{2} m v^2 = \tfrac{1}{2}(m r^2)\omega^2 .$$

We see that the kinetic energy is (1/2) times a constant factor (since r is constant for circular motion) times ω^2. The constant factor is called the **moment of inertia**, denoted by I. In the present rather trivial case of a single particle

$$K = \tfrac{1}{2} I \omega^2 \text{ where } I = m r^2 .$$

We will extend the definition of I to a system of particles below.

Some authors prefer the more suggestive term "rotational inertia" rather than "moment of inertia."

Rotation of a rigid body about a fixed axis

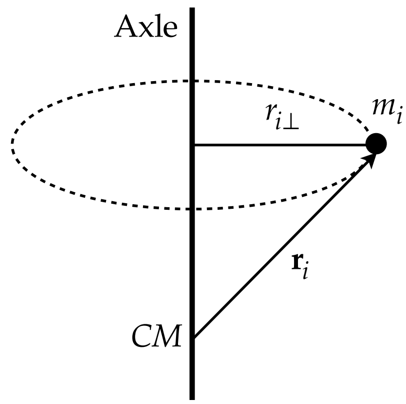
Because the particles in a rigid body maintain a fixed spatial relationship to each other, the only allowed motion relative to the CM is rotation. The CM can of course move in an arbitrary way, subject to the effects of the total force.

The most general motion of a rigid body is translational motion of the CM plus rotation about an axis through the CM.

We are interested primarily in the rotational motion. To simplify the discussion, we will assume at first that the rigid body is mounted on a fixed axle passing through the CM.

If the body is not mounted on an axle, the axis of rotation can itself move about in complicated ways. An example is a badly thrown forward pass in football, which wobbles. That kind of case is too difficult to be treated here, so we mainly consider bodies mounted on axles.

We choose the CM as the origin. The body rotates with angular speed ω around the axle shown. Also shown is one particle of the rigid body, and the circle it describes as the body rotates. Its position vector is $\mathbf{r}_i$. The component of $\mathbf{r}_i$ perpendicular to the axis, which we call $\mathbf{r}_{i\perp}$, is the radius of the circle. The linear speed of the particle is $\omega r_{i\perp}$, so its kinetic energy of rotation of this particle around the axle is thus



$$K_i = \tfrac{1}{2} m_i r_{i\perp}^2 \omega^2 .$$

Summing over all the particles in the rigid body, we get the total kinetic energy of rotation:

$$K_{rot} = \tfrac{1}{2}\left(\sum_i m_i {r_{i\perp}}^2 \right) \omega^2 .$$

The quantity in ( ) is by definition the moment of inertia of the whole system:

Moment of inertia of a system	$I = \sum_i m_i r_{i\perp}^2$

This quantity is a measure of the inertia of the body with respect to rotational motion.

It is important to remember that $r_{i\perp}$ is the *perpendicular* distance from the particle to the axis of rotation, i.e., the radius of the circle followed by the particle as the body rotates.

One uses this sum to calculate I only if the body consists of a small number of discrete particles. For macroscopic solid objects (where the number of atoms is very large) one treats the mass distribution as a continuum, and the sum becomes an integral.

Strictly speaking, this would be an approximation, but one uses it in dealing with macroscopic systems. Interestingly, until near the end of the 19th century many scientists believed matter really was distributed continuously in space, while atoms were a useful model, but might not actually exist.

Because of the dependence of I on $r_{i\perp}^2$, bodies with more mass far from the axis have larger moments of inertia. For example, a uniform circular hoop of mass M and radius R has $I = MR^2$, while for a uniform sphere of the same mass and radius $I = \frac{2}{5}MR^2$.

A table of I for various symmetric bodies is given in most texts. The formulas for such bodies will be given on quizzes and exams.

We have shown here that the *rotational* part of the kinetic energy of a rigid body is

Kinetic energy of rotation	$K_{rot} = \frac{1}{2}I\omega^2$

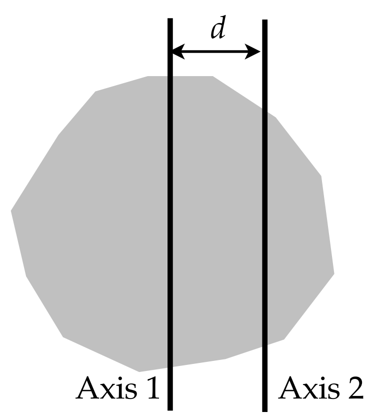
This is a very important formula, with many applications.

The parallel axis theorem

The value of I depends on what axis of rotation is used. There is a simple theorem relating I for two different axes, provided that:

- The axes are parallel to each other.
- One of the axes passes through the CM of the body.

Shown is such a case, where Axis 1 passes through the CM.



We assume that the moment of inertia about Axis 1, which we call I_{CM}, is known. We wish to find the moment of inertia about Axis 2, which we call I.

If the body rotates with angular speed ω about Axis 2, its kinetic energy is

$$K = \frac{1}{2}I\omega^2.$$

As each particle makes a complete revolution about Axis 2 it also makes a complete revolutions about Axis 1, so the particles move at the same angular speed about both axes. The kinetic energy of the rotation about Axis 1 is

$$K_{CM} = \tfrac{1}{2} I_{CM} \omega^2 .$$

During each complete rotation of the body about Axis 2 the CM revolves once in a circle of radius d. The linear speed of this circular motion is ωd.

Now the total kinetic energy of rotation about Axis 2 (by the general theorem) is equal to the kinetic energy of motion of a single particle with the total mass, moving with the CM, plus the energy of motion about the CM. Thus we have

$$\tfrac{1}{2} I \omega^2 = \tfrac{1}{2} M (d\omega)^2 + \tfrac{1}{2} I_{CM} \omega^2 .$$

Dividing out the common factor $\tfrac{1}{2}\omega^2$ we find a useful theorem:

Parallel axis theorem	$I = I_{CM} + Md^2$

We see that of all parallel axes the one through the CM has the smallest value of I.